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COMMENT

Exact solutions to non-linear chiral field equations

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Abstract. Two types of exact solutions for the non-linear field equations for the chiral invariant model of pion dynamics are presented here. One of them is a generalisation of the solutions obtained by Charap. This more general solution includes the soliton solution as a special case.

1. Introduction

Under tangential parametrisation (Charap 1973) the field equations for the chiral invariant model of pion dynamics take the form (Charap 1976)

$$\Box\phi = \eta^{\mu\nu}\frac{\partial\phi}{\partial x}\frac{\partial\beta}{\partial x^{\nu}}, \qquad \Box\psi = \eta^{\mu\nu}\frac{\partial\psi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}}, \qquad \Box\chi = \eta^{\mu\nu}\frac{\partial\chi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{1.1}$$

where

$$\eta^{\mu\nu} = \begin{cases} 0 & \text{for } \mu \neq \nu, \\ 1 & \text{for } \mu = \nu \neq 4, \\ -1 & \text{for } \mu = \nu = 4, \end{cases}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2),$$

f_π is a constant, and ϕ, ψ and χ are the pion field triplet. The Lagrangian is given by

$$L = \tfrac{1}{2}(g_{11}\,\partial_\mu\phi\,\partial^\mu\phi + g_{22}\,\partial_\mu\psi\,\partial^\mu\psi + g_{33}\,\partial_\mu\chi\,\partial^\mu\chi + 2g_{12}\,\partial_\mu\phi\,\partial^\mu\psi + 2g_{13}\,\partial_\mu\phi\,\partial^\mu\chi + 2g_{23}\,\partial_\mu\psi\,\partial^\mu\chi) \tag{1.2}$$

where the g_{ij} are such that Γ^l_{ij}, the Christoffel symbols, take the form

$$\Gamma^l_{ij} = -(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)^{-1}(\delta^l_j\phi_i + \delta^l_i\phi_j). \tag{1.3}$$

In (1.3), ϕ_1, ϕ_2 and ϕ_3 represent respectively ϕ, ψ and χ.

Charap obtained solutions of (1.1) under the assumption that ϕ, ψ and χ are all functions of $k_1x^1 + k_2x^2 + k_3x^3 + k_4x^4$ where k_μ is any four-vector. But such solutions give non-vanishing derivatives of ϕ, ψ, χ at infinity and hence cannot represent soliton solutions. However, in the present note, we shall see that equations (1.1) and (1.2) can be integrated under a weaker assumption that there exists some function u, such that ϕ, ψ and χ are functions of u. These solutions include the soliton solution as a special case. Another class of solutions of (1.1) and (1.2) will also be presented here.

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2. Generalisation of Charap's solution

Assume that there exists a function u such that

$$\phi=\phi(u), \qquad \psi=\psi(u), \qquad \chi=\chi(u); \tag{2.1}$$

(1.1) then reduces to:

$$\begin{aligned}
&(\phi_{uu}-\phi_u\beta_u)(u_1^2+u_2^2+u_3^2-u_4^2)+\phi_u\Box u=0,\\
&(\psi_{uu}-\psi_u\beta_u)(u_1^2+u_2^2+u_3^2-u_4^2)+\psi_u\Box u=0,\\
&(\chi_{uu}-\chi_u\beta_u)(u_1^2+u_2^2+u_3^2-u_4^2)+\chi_u\Box u=0.
\end{aligned} \tag{2.2}$$

$\phi_u\equiv \mathrm{d}\phi/\mathrm{d}u$ and so on; $u_1\equiv\partial u/\partial x^1$, and so on. From (2.2), either

$$\frac{\phi_{uu}-\phi_u\beta_u}{\phi_u}=\frac{\psi_{uu}-\psi_u\beta_u}{\psi_u}=\frac{\chi_{uu}-\chi_u\beta_u}{\chi_u} \tag{2.3}$$

or

$$u_1^2+u_2^2+u_3^2-u_4^2=0 \qquad \text{and} \qquad \Box u=0. \tag{2.4}$$

2.1. *Case 1: (2.3) is true*

(2.3) can be simplified to

$$\frac{\phi_{uu}}{\phi_u}=\frac{\psi_{uu}}{\psi_u}=\frac{\chi_{uu}}{\chi_u},$$

which, on integration, shows that ϕ and ψ are linearly related and so are ψ and χ. Then one can choose u, consistent with (2.1) such that

$$\begin{aligned}
&\phi=au+b,\\
&\psi=cu+d,\\
&\chi=eu+f,
\end{aligned} \tag{2.5}$$

where a, b, c, d, e and f are constants.

Putting (2.5) into (1.1) and (1.2), we see that (2.5) satisfies (1.1) if u is given by

$$\zeta=\int\frac{\mathrm{d}u}{(a^2+c^2+e^2)u^2+2u(ab+cd+ef)+(b^2+d^2+f^2+f_\pi^2)}, \tag{2.6}$$

where ζ satisfies

$$\zeta_{11}+\zeta_{22}+\zeta_{33}-\zeta_{44}=0. \tag{2.7}$$

(2.6) can readily be integrated to give u in terms of ζ and (2.7) has the known solutions.

2.2. *Case 2: (2.4) is true*

(2.4) is satisfied by

$$u=G(k_1x^1+k_2x^2+k_3x^3+k_4x^4), \tag{2.8}$$

where

$$k_1^2+k_2^2+k_3^2-k_4^2=0, \tag{2.9}$$

k_1, k_2, k_3 and k_4 being constants, and G being an arbitrary function.

Thus, when u is given by (2.8) and (2.9), (2.1) gives a class of solutions of (1.1). However, other solutions of (2.4), if found, will also give solutions of (1.1) of the form (2.1).

3. Other types of solutions

Assume

$$\begin{aligned}\phi &= \phi(x^1, x^2, x^3-x^4),\\ \psi &= \psi(x^1, x^2, x^3-x^4),\\ \chi &= \chi(x^1, x^2, x^3-x^4).\end{aligned} \tag{3.1}$$

When ϕ, ψ and χ are of the form (3.1), equations (1.1) reduce to

$$\begin{aligned}\phi_{11}+\phi_{22} &= \phi_1\beta_1+\phi_2\beta_2,\\ \psi_{11}+\psi_{22} &= \psi_1\beta_1+\psi_2\beta_2,\\ \chi_{11}+\chi_{22} &= \chi_1\beta_1+\chi_2\beta_2.\end{aligned} \tag{3.2}$$

It is to be noted that although ϕ, ψ and χ are functions of three variables x^1, x^2, x^3-x^4, (3.2) involves two variables, x^1 and x^2, only.

Although the present author has failed to solve (3.2) completely, particular solutions of (3.2) can be picked up easily. Two such solutions are presented here.

3.1. Solution 1

Assume

$$\phi = \phi(\alpha, x^3-x^4), \qquad \psi = \psi(\alpha, x^3-x^4), \qquad \chi = \chi(\alpha, x^3-x^4), \tag{3.3}$$

where

$$\alpha = \alpha(x^1, x^2, x^3-x^4). \tag{3.4}$$

Proceeding as in § 2, but noting that $\alpha_1^2+\alpha_2^2 \geq 0$, we get the following solution:

$$\phi = g\alpha + h, \qquad \psi = i\alpha + j, \qquad \chi = k\alpha + l, \tag{3.5}$$

where g, h, i, j, k and l are functions of (x^3-x^4) and α is given by

$$\xi = \int \frac{\mathrm{d}\alpha}{(g^2+i^2+k^2)\alpha^2+2\alpha(gh+ij+kl)+(h^2+j^2+l^2+f_\pi^2)} \tag{3.6}$$

and ξ satisfies

$$\xi_{11}+\xi_{22}=0. \tag{3.7}$$

As before, the right-hand side of (3.6) can be integrated to express α in terms of ξ.

3.2. *Solution 2*

Assume

$$\begin{aligned}\phi &= \gamma(x^2, x^3-x^4)p(x^1, x^3-x^4),\\ \psi &= \gamma(x^2, x^3-x^4)q(x^1, x^3-x^4),\\ \chi &= \gamma(x^2, x^3-x^4)r(x^1, x^3-x^4),\end{aligned} \tag{3.8}$$

such that

$$p^2+q^2+r^2=1. \tag{3.9}$$

Putting (3.8) and (3.9) into (3.2) and using separation of variables

$$\frac{p_{11}}{p}=\frac{q_{11}}{q}=\frac{r_{11}}{r}=m \tag{3.10}$$

and

$$\gamma_{22}+m\gamma=\frac{2\gamma_2^2}{f_\pi^2+\gamma^2}, \tag{3.11}$$

where m is a function of (x^3-x^4).

(3.10) can easily be integrated, but p, q and r must also satisfy (3.9). From this it follows that m must be negative and we have the following solutions for p, q and r:

$$\begin{aligned}&p=A\cos nx^1+B\sin nx^1, \qquad q=C\cos nx^1+D\sin nx^1,\\ &r=E\cos nx^1+F\sin nx^1,\end{aligned} \tag{3.12}$$

where n, A, B, C, D, E and F are functions of (x^3-x^4) satisfying

$$A^2+C^2+E^2=1, \qquad B^2+D^2+F^2=1, \qquad AB+CD+EF=0 \tag{3.13}$$

and $m=-n^2$.

(3.11) can be integrated to give

$$\int\frac{\mathrm{d}\gamma}{\exp[2/(f_\pi^2+\gamma^2)]\{H+2n^2\int\gamma\exp[-4/(f_\pi^2+\gamma^2)]\,\mathrm{d}\gamma\}^{1/2}}=\pm x^2+I, \tag{3.14}$$

H and I being arbitrary functions of (x^3-x^4).

4. Conclusion

In summing up, we note that case 2 in § 2.2 can, by a suitable choice of coordinates, be expressed as a special case of solution 1 in § 3.1. Thus we have only the following three solutions of (1.1) here.

4.1. *Solution 1*

ϕ, ψ and χ are given by (3.5), where g, h, i, j, k and l are functions of (x^3-x^4), and α is given by (3.6) and (3.7).

4.2. Solution 2

ϕ, ψ and χ are given by (3.8) where p, q and r are given by (3.12), γ is given by (3.14), n, A, B, C, D, E, F, H and I are arbitrary functions of $(x^3 - x^4)$.

4.3. Solution 3

ϕ, ψ and χ are given by (2.5), a, b, c, d, e and f being constants, and u is given by (2.6) and (2.7).

Of these three types of solutions, solutions 1 and 2 have non-vanishing derivatives of ϕ, ψ and χ at infinity and hence cannot represent soliton solutions. However, solution 3 includes the soliton solution as a special case as can be seen as follows.

Integrating the right-hand side of (2.6), we get

$$U = -Q + P \tan(\zeta RA)$$

where

$$P = \frac{(b^2 + d^2 + f^2 + f_\pi^2)/(a^2 + c^2 + d^2) - (ab + cd + ef)^2}{(a^2 + c^2 + d^2)^2},$$

$$Q = \frac{ab + cd + ef}{a^2 + c^2 + d^2}, \qquad R = a^2 + c^2 + d^2, \tag{4.1}$$

a, b, c, d, e and f being the same constants as in (2.6) and ζ is a solution of (2.7), ϕ, ψ and χ are given by (2.5).

If the particular solution of (2.7) that we pick here is

$$\zeta = \frac{\sin r}{r} \cos t \tag{4.2}$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $t = x^4$ and R and A are such that $\max(\sin r/r) < \pi/2RA$. Obviously $\sin r/r$ is bounded, so it is easy to see from (2.5), (4.1) and (4.2) that ϕ, ψ and χ and their derivatives are bounded everywhere and the derivatives of ϕ, ψ and χ vanish sufficiently fast at infinity for the integral of Hamiltonian density over entire space to be finite. Thus we get a soliton solution.

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